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Light scattering in anisotropic stratified media

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We determine exact formulas, in the Born approximation, for the scattering cross sections of the electromagnetic field propagating in a scattering anisotropic stratified medium having an arbitrary variation of the average dielectric tensor along the longitudinal direction and bounded by outermost isotropic homogeneous media. The approach is based on a suitable generalization of the Berreman's equations describing the propagation of the electromagnetic field in a transparent anisotropic stratified medium. A few particularly interesting examples are worked out in order to show, both analytically and numerically, how the previously known results are recovered under suitable approximations, with particular attention to the subtleties connected to the birefringence of the scattering medium and the transformations of the solid angles between the scattering and detection regions: it is shown that these are automatically included in a straightforward manner in the new formalism. The resulting exact equations can be numerically evaluated in a very easy way, as they involve very simple algebraic matrix manipulations.

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I. INTRODUCTION

The light scattering techniques are a very useful tool for the investigation of various different physical systems. In the case of nematic liquid crystals, for example, the Rayleigh light scattering can be used to determine all the elastic and viscosity coefficients [1–4].

The existing theories are based on a number of simplifying assumptions as the single-scattering, or Born, approximation and the homogeneity of the scattering medium. The simplest situation arises when the unperturbed scattering medium is isotropic, with the same index of refraction as the surrounding region in which the incident light propagates and the scattered radiation is detected. In this case exact and very simple analytical formulas, in the Born approximation, are available [5]. In the actual experiments, however, usually the average scattering medium, even when isotropic, has a different index of refraction with respect to the external regions. In this case the refraction at the boundaries, in addition to transmission losses, introduces a modification of the solid angle of the scattered beam, an effect that is not trivial at all, as discussed in [6]. The situation is even more difficult in the case of an anisotropic medium, due to the noncollinearity between the wave vector and the Poynting vector. The mere calculation of the asymptotic behavior of the scattered field inside a birefringent medium, neglecting the boundaries, requires a careful analysis [6–11].

The solutions to the scattering problem that have been so far given are all based on a Green's propagator approach, and therefore require special considerations in order to take into account the stratification of the scattering environment: in [12], where the Rayleigh light scattering method for determining the ratios between the elastic constants of a nematic liquid crystal is reviewed, the approximate expressions obtained by neglecting the optical anisotropy of the liquid crystal and the corrections that must be introduced to take into account the birefringence of the medium and the presence of the boundaries are discussed. On the other hand, it often happens that various boundaries are present, and sometimes the unperturbed scattering medium itself has a dielectric tensor varying along the longitudinal direction [13].

In order to overcome all the difficulties in getting reasonably approximate and tractable expressions in these situations, we present here a very different approach to the problem, that allows one to obtain exact expressions for the scattering cross sections in the case of a scattering anisotropic medium with an arbitrary variation of the average dielectric tensor along the stratification direction, embedded in an arbitrarily stratified dielectric transparent medium whose outermost parts are isotropic. The formalism is based on an expansion of the scattered field in Fourier components in the transverse directions and a generalization of the equations describing the propagation of the transverse components of the electromagnetic field in a transparent stratified medium [14]: it automatically incorporates all the boundary and birefringence corrections in a very simple and compact manner.

In Sec. II we determine the propagation equations, in the Born approximation, for the transverse components of the incident and scattered electromagnetic field in a birefringent scattering medium, having an arbitrary variation of the average dielectric tensor in the longitudinal direction, by generalizing the Berreman's equations for transparent anisotropic stratified media. The asymptotic behavior of the scattered field is evaluated in Sec. III and used to obtain the scattered power per unit solid angle in the external isotropic regions. This is employed in Sec. IV to derive the differential scattering cross sections, for an unperturbed homogeneous anisotropic scattering slab sandwiched between identical external isotropic dielectric media, by imposing suitable boundary conditions on the incident and scattered fields. The resulting equations are analytically compared with the already known results in Sec. V, by considering the special cases of a scattering isotropic medium having the same unperturbed index of refraction of the external media, and of a uniaxial scattering slab represented by a planarly aligned nematic liquid crystal cell: for the latter case approximate analytical expressions are derived from the exact theory, by neglecting the reflections and transmissions at the boundaries, and shown to coincide with the approximate expressions obtained with the Green's propagator approach and the transformations of solid angles between the scattering and the external media. A numerical comparison between the exact and approximate scattering cross sections is presented and discussed. In Sec. VI we obtain a general formula giving the scattering cross sections in an anisotropic medium having an arbitrary stepwise or continuous variation of the unperturbed dielectric tensor. Finally, in Sec. VII, we summarize our results.

II. FIELD EQUATIONS

Let us consider a stratified medium consisting of an anisotropic scattering slab, confined between the planes $z = 0$ and $z = d$ of a Cartesian coordinate system (x, y, z), sandwiched between external dielectric homogeneous media having index of refraction n_1. Supposing that the fluctuations of the dielectric tensor giving rise to light scattering are much slower with respect to the optical frequency of the incident monochromatic beam, we can neglect the frequency shifts of the scattered light and thus write the total electromagnetic field as

$$\vec{\mathcal{E}}(\vec{r}, t) = \vec{E}(\vec{r}) \exp(-i\omega t) + \text{c.c.} , \quad (2.1a)$$

$$\vec{\mathcal{H}}(\vec{r}, t) = \vec{H}(\vec{r}) \exp(-i\omega t) + \text{c.c.} , \quad (2.1b)$$

where c.c. indicates complex conjugate. We write the components $\epsilon_{\alpha\beta}$ $(\alpha, \beta = x, y, z)$ of the complex relative dielectric tensor of the stratified medium as

$$\epsilon_{\alpha\beta}(\vec{r}, t) = \eta_{\alpha\beta}(z) + \delta\epsilon_{\alpha\beta}(\vec{r}, t) , \quad (2.2)$$

where $\eta_{\alpha\beta}(z)$ represents the relative dielectric tensor of the unperturbed structure and thus depends only on the transverse coordinate z, while $\delta\epsilon_{\alpha\beta}(\vec{r}, t)$ is a small fluctuation responsible for the light scattering that is different from zero only for $0 \leq z \leq d$; we decompose the latter in a bidimensional Fourier integral in the transverse plane (x, y) according to

$$\delta\epsilon_{\alpha\beta}(\vec{r}, t) = \int \delta\tilde{\epsilon}_{\alpha\beta}(p, q, z, t) \exp[ik_0(px + qy)]\, dp\, dq , \quad (2.3a)$$

$$\delta\tilde{\epsilon}_{\alpha\beta}(p, q, z, t) = \left(\frac{k_0}{2\pi}\right)^2 \int \delta\epsilon_{\alpha\beta}(\vec{r}, t) \times \exp[-ik_0(px + qy)]\, dx\, dy , \quad (2.3b)$$

$k_0 = \omega/c$ being the modulus of the vacuum wave vector of the incident field.

The total electromagnetic field propagating in the stratified medium is now decomposed in an incident (initial) and a scattered (final) part

$$\vec{E} = \vec{E}_i + \vec{E}_f , \quad (2.4a)$$

$$\vec{H} = \vec{H}_i + \vec{H}_f . \quad (2.4b)$$

For the translational invariance of the unperturbed structure in the transverse plane, we can look for a plane wave incident field in the form

$$\vec{E}_i = Z_0^{1/2}\, \vec{e}_i(z) \exp[ik_0(p_i x + q_i y)] , \quad (2.5a)$$

$$\vec{H}_i = Z_0^{-1/2}\, \vec{h}_i(z) \exp[ik_0(p_i x + q_i y)] , \quad (2.5b)$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the vacuum characteristic impedance and

$$p_i = n_1 \sin\vartheta_i \cos\varphi_i , \quad (2.6a)$$

$$q_i = n_1 \sin\vartheta_i \sin\varphi_i , \quad (2.6b)$$

ϑ_i and φ_i being the polar angles of the incident beam, measured in the homogeneous incident medium having index of refraction n_1. The scattered field, in turn, can be decomposed in Fourier integral in the transverse directions

$$\vec{E}_f = Z_0^{1/2} \int \vec{e}_f(p, q, z) \exp[ik_0(px + qy)]\, dp\, dq , \quad (2.7a)$$

$$\vec{H}_f = Z_0^{-1/2} \int \vec{h}_f(p, q, z) \exp[ik_0(px + qy)]\, dp\, dq . \quad (2.7b)$$

By inserting Eqs. (2.1)–(2.7) into the Maxwell equations, in the Born approximation, i.e., neglecting the products between the dielectric tensor fluctuations and the scattered fields, that give rise to multiple scattering, and algebraically expressing the z components of the fields as a function of the transverse components [14,15], we get the following set of equations for the transverse components of the fields:

$$\frac{d\psi_i}{dz} = ik_0 D_i \psi_i , \quad (2.8a)$$

$$\frac{d\psi_f}{dz} = ik_0 D_f \psi_f + ik_0 G \psi_i , \quad (2.8b)$$

that represents a generalization of the Berreman's equations for stratified homogeneous anisotropic media [14].

In Eq. (2.8a) ψ_i is the column vector of the normalized transverse components of the incident electromagnetic field

$$\psi_i \equiv \psi_i(z) = \begin{pmatrix} e_{ix} \\ h_{iy} \\ e_{iy} \\ -h_{ix} \end{pmatrix} , \quad (2.9)$$

with $e_{ix} = \vec{e}_i \cdot \hat{\mathbf{x}}$, $e_{iy} = \vec{e}_i \cdot \hat{\mathbf{y}}$, $h_{ix} = \vec{h}_i \cdot \hat{\mathbf{x}}$, $h_{iy} = \vec{h}_i \cdot \hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ ($\hat{\mathbf{y}}$) is the unit vector in the x (y) direction; the free evolution of the incident plane wave in the unperturbed structure is governed by the 4×4 matrix D_i that is given explicitly by [14,15]

$$D_i = \begin{pmatrix} -p_i\eta_{zz}^{-1}\eta_{zx} & 1-p_i^2\eta_{zz}^{-1} & -p_i\eta_{zz}^{-1}\eta_{zy} & -p_iq_i\eta_{zz}^{-1} \\ \eta_{xx}-\eta_{zz}^{-1}\eta_{xz}\eta_{zx}-q_i^2 & -p_i\eta_{zz}^{-1}\eta_{xz} & \eta_{xy}-\eta_{zz}^{-1}\eta_{xz}\eta_{zy}+p_iq_i & -q_i\eta_{zz}^{-1}\eta_{xz} \\ -q_i\eta_{zz}^{-1}\eta_{zx} & -p_iq_i\eta_{zz}^{-1} & -q_i\eta_{zz}^{-1}\eta_{zy} & 1-q_i^2\eta_{zz}^{-1} \\ \eta_{yx}-\eta_{zz}^{-1}\eta_{yz}\eta_{zx}+p_iq_i & -p_i\eta_{zz}^{-1}\eta_{yz} & \eta_{yy}-\eta_{zz}^{-1}\eta_{yz}\eta_{zy}-p_i^2 & -q_i\eta_{zz}^{-1}\eta_{yz} \end{pmatrix} . \quad (2.10)$$

Similarly, in Eq. (2.8b), ψ_f is the column vector containing the transverse components of the scattered field for a given (p,q) Fourier component, being the various Fourier components independent one of each other,

$$\psi_f \equiv \psi_f(p,q,z) = \begin{pmatrix} e_{fx} \\ h_{fy} \\ e_{fy} \\ -h_{fx} \end{pmatrix} , \quad (2.11)$$

with $e_{fx} = \vec{e}_f(p,q,z)\cdot\hat{\mathbf{x}}$, $e_{fy} = \vec{e}_f(p,q,z)\cdot\hat{\mathbf{y}}$, $h_{fx} = \vec{h}_f(p,q,z)\cdot\hat{\mathbf{x}}$, and $h_{fy} = \vec{h}_f(p,q,z)\cdot\hat{\mathbf{y}}$. The matrix D_f describes the free evolution of each transverse Fourier component of the scattered field in the unperturbed structure, and is given by the same expression (2.10) with p and q replacing p_i and q_i, respectively. The coupling between the incident and the scattered field is expressed by the matrix G in the right hand side of Eq. (2.8b) whose elements are given by

$$G_{11} = p\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zx}\right) , \quad (2.12a)$$

$$G_{12} = pp_i\eta_{zz}^{-2}\delta\tilde{\epsilon}_{zz} , \quad (2.12b)$$

$$G_{13} = p\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zy}\right) , \quad (2.12c)$$

$$G_{14} = pq_i\eta_{zz}^{-2}\delta\tilde{\epsilon}_{zz} , \quad (2.12d)$$

$$G_{21} = \eta_{zz}^{-1}\eta_{xz}\left(\eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zx}\right) + \delta\tilde{\epsilon}_{xx} - \eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{xz} , \quad (2.12e)$$

$$G_{22} = p_i\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{xz}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{xz}\right) , \quad (2.12f)$$

$$G_{23} = \eta_{zz}^{-1}\eta_{xz}\left(\eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zy}\right) + \delta\tilde{\epsilon}_{xy} - \eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{xz} , \quad (2.12g)$$

$$G_{24} = q_i\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{xz}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{xz}\right) , \quad (2.12h)$$

$$G_{31} = q\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zx}\right) , \quad (2.12i)$$

$$G_{32} = qp_i\eta_{zz}^{-2}\delta\tilde{\epsilon}_{zz} , \quad (2.12j)$$

$$G_{33} = q\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zy}\right) , \quad (2.12k)$$

$$G_{34} = qq_i\eta_{zz}^{-2}\delta\tilde{\epsilon}_{zz} , \quad (2.12l)$$

$$G_{41} = \eta_{zz}^{-1}\eta_{yz}\left(\eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zx}\right) + \delta\tilde{\epsilon}_{yx} - \eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{yz} , \quad (2.12m)$$

$$G_{42} = p_i\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{yz}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{yz}\right) , \quad (2.12n)$$

$$G_{43} = \eta_{zz}^{-1}\eta_{yz}\left(\eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zy}\right) + \delta\tilde{\epsilon}_{yy} - \eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{yz} , \quad (2.12o)$$

$$G_{44} = q_i\eta_{zz}^{-1}\left(\eta_{zz}^{-1}\eta_{yz}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{yz}\right) , \quad (2.12p)$$

where, for $\alpha,\beta = x,y,z$, $\delta\tilde{\epsilon}_{\alpha\beta} \equiv \delta\tilde{\epsilon}_{\alpha\beta}(p-p_i,q-q_i,z,t)$. Therefore the coupling matrix G depends linearly on the transverse Fourier transforms of the fluctuations of the dielectric tensor computed for a transverse wave vector corresponding to the scattering momentum.

The longitudinal components of the scattered field can be expressed as a function of the transverse components of the scattered and incident field as

$$\begin{pmatrix} e_{fz} \\ h_{fz} \end{pmatrix} = C_f\,\psi_f + C_{fi}\,\psi_i , \quad (2.13)$$

with

$$C_f = \begin{pmatrix} -\eta_{zz}^{-1}\eta_{zx} & -p\eta_{zz}^{-1} & -\eta_{zz}^{-1}\eta_{zy} & -q\eta_{zz}^{-1} \\ -q & 0 & p & 0 \end{pmatrix} , \quad (2.14)$$

and

$$C_{fi} = \begin{pmatrix} \eta_{zz}^{-1}[\eta_{zz}^{-1}\eta_{zx}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zx}] & p_0\eta_{zz}^{-2}\delta\tilde{\epsilon}_{zz} & \eta_{zz}^{-1}[\eta_{zz}^{-1}\eta_{zy}\delta\tilde{\epsilon}_{zz} - \delta\tilde{\epsilon}_{zy}] & q_0\eta_{zz}^{-2}\delta\tilde{\epsilon}_{zz} \\ 0 & 0 & 0 & 0 \end{pmatrix} . \quad (2.15)$$

Similarly, for the longitudinal components of the incident field, we have

$$\begin{pmatrix} e_{iz} \\ h_{iz} \end{pmatrix} = C_i\,\psi_i , \quad (2.16)$$

where the 2×4 matrix C_i is given by the same expression (2.14) but with p_i and q_i replacing p and q, respectively.

III. ASYMPTOTIC BEHAVIOR

In order to compute the scattered power per unit solid angle, we must determine the asymptotic behavior of the scattered field (2.7) for $k_0 r \gg 1$, where $r = \sqrt{x^2+y^2+z^2}$.

Let us first consider the forward scattered power, i.e., the asymptotic solutions in the region $z > d$. In this zone the coupling matrix G is zero, since no scattering occurs, and the scattered field freely propagates in a homogeneous medium having index of refraction n_1. The solutions of Eq. (2.8b) are then given by a linear superposition of the four proper electromagnetic waves $\psi_f^{(n)}$ $(n = 1,\ldots,4)$ that are eigenvectors of the z-independent propagation matrix D_f obtained by setting $\eta_{\alpha\beta} = n_1^2\,\delta_{\alpha\beta}$, $\delta_{\alpha\beta}$ being the Kronecker's δ,

$$\psi_f(p,q,z) = \sum_{n=1}^{4} c_n(p,q)\,\psi_f^{(n)}(p,q)\exp[ik_0\gamma_n(p,q)z] , \quad (3.1)$$

where the $c_n(p,q)$ are z-independent coefficients and the $\gamma_n(p,q)$ are the eigenvalues of the matrix D_f associated to the eigenvectors $\psi_f^{(n)}(p,q)$. As it is evident, the proper waves are four plane waves with the same value of the transverse wave vector, two forwards and two backwards propagating, each with two polarization states orthogonal to the propagation direction: for nonevanescent waves, when properly normalized, they form a complete orthonormal set in the sense specified in [15], a property that holds true for all nonabsorbing media. Precisely, by choosing linear transverse electric (TE) and transverse magnetic (TM) polarizations, and normalizing the waves in such a way that they have a z component of the time-averaged Poynting vector equal to 1 (−1) for the forwards (backwards) solutions, we can put

$$\psi_f^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} s/u \\ su \\ -c/u \\ -cu \end{pmatrix}, \quad \psi_f^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} cu/n_1 \\ cn_1/u \\ su/n_1 \\ sn_1/u \end{pmatrix}, \tag{3.2a}$$

$$\psi_f^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} s/u \\ -su \\ -c/u \\ cu \end{pmatrix}, \quad \psi_f^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -cu/n_1 \\ cn_1/u \\ -su/n_1 \\ sn_1/u \end{pmatrix}, \tag{3.2b}$$

with

$$c = \frac{p}{\sqrt{p^2+q^2}}, \quad s = \frac{q}{\sqrt{p^2+q^2}},$$
$$u = \sqrt[4]{n_1^2 - (p^2+q^2)}\,. \tag{3.3}$$

With these conventions $\psi_f^{(1)}$ ($\psi_f^{(2)}$) represents a forward TE (TM) wave, while $\psi_f^{(3)}$ ($\psi_f^{(4)}$) a backward TE (TM) wave. The corresponding eigenvalues are given by

$$\gamma_1(p,q) = \gamma_2(p,q) \equiv \gamma_+(p,q) = \sqrt{n_1^2 - (p^2+q^2)}\,, \tag{3.4a}$$

$$\gamma_3(p,q) = \gamma_4(p,q) \equiv \gamma_-(p,q) = -\sqrt{n_1^2 - (p^2+q^2)}\,, \tag{3.4b}$$

as easily follows also from the fact that the modulus of the wave vector of each plane wave must be equal to $k_0 n_1$. Therefore the total electromagnetic field generated by all the progressive eigenvectors having a given polarization state reads

$$\vec{E}_f(\vec{r}) = Z_0^{1/2}\int c_n(p,q)\vec{e}_f^{\,(n)}(p,q) \times \exp[ik_0(px+qy+\gamma_+(p,q)z)]\,dp\,dq\,, \tag{3.5a}$$

$$\vec{H}_f(\vec{r}) = Z_0^{-1/2}\int c_n(p,q)\vec{h}_f^{(n)}(p,q) \times \exp[ik_0(px+qy+\gamma_+(p,q)z)]\,dp\,dq\,, \tag{3.5b}$$

where $\vec{e}_f^{\,(n)}$ ($\vec{h}_f^{(n)}$) is the total electric (magnetic) field corresponding to the transverse components $\psi_f^{(n)}$ and $n=1$ ($n=2$) for the TE (TM) polarization.

The asymptotic behavior of (3.5) is easily evaluated with the help of the stationary phase method [16,17], that amounts to expanding the rapidly oscillating phase factor in the complex exponentials up to second order in a Taylor series about its stationary point ($p = p_f, q = q_f$) and to substitute the slowly varying amplitudes with their values at the stationary point. For $z > 0$ we easily obtain

$$\vec{E}_f(\vec{r}) = \left(\frac{2\pi}{ik_0 r}\right) Z_0^{1/2} n_1 \cos\vartheta_f\, c_n(p_f,q_f) \times\vec{e}_f^{\,(n)}(p_f,q_f)\exp(ik_0 n_1 r)\,, \tag{3.6a}$$

$$\vec{H}_f(\vec{r}) = \left(\frac{2\pi}{ik_0 r}\right) Z_0^{-1/2} n_1 \cos\vartheta_f\, c_n(p_f,q_f) \times\vec{h}_f^{(n)}(p_f,q_f)\exp(ik_0 n_1 r)\,, \tag{3.6b}$$

with

$$p_f = n_1 \sin\vartheta_f \cos\varphi_f\,, \tag{3.7a}$$

$$q_f = n_1 \sin\vartheta_f \sin\varphi_f\,, \tag{3.7b}$$

being ϑ_f and φ_f the polar angles of the scattering direction. It is easily verified that, as it is obvious, the backward propagating Fourier components give rise to backward propagating spherical waves, and must be therefore absent for $z > d$.

The scattered power per unit solid angle for a given polarization is now given by

$$\frac{dP_f}{d\Omega} = r^2\left(\vec{S}\cdot\hat{\mathbf{r}}\right)\,, \tag{3.8}$$

where $\hat{\mathbf{r}} = \vec{r}/r$ is the unit vector parallel to the scattering direction and $\vec{S}$ is the time-averaged Poynting vector

$$\vec{S} = \vec{E}_f \times \vec{H}_f^* + \text{c.c.}\,. \tag{3.9}$$

According to (3.6) and taking into account that the proper waves have been normalized to unit z component of the time-averaged Poynting vector, that in an isotropic medium is directed along the wave vector, we then immediately obtain

$$\frac{dP_f}{d\Omega} = \lambda^2 n_1^2 \cos\vartheta_f |c_n(p_f,q_f)|^2\,, \tag{3.10}$$

$\lambda = 2\pi/k_0$ being the vacuum wavelength of the incident radiation.

In the region $z < 0$, instead, only the backward propagating Fourier components must be present in order to give rise to a spherical wave propagating towards infinity. The corresponding scattered power per unit solid angle for a given polarization is again given by Eq. (3.10) with $n = 3, 4$, but for a scattering direction characterized by the polar angle $\pi - \vartheta_f$.

IV. BOUNDARY CONDITIONS AND SCATTERING CROSS SECTIONS

Let us now consider the propagation of the incident and scattered fields and the related boundary conditions.

According to Eq. (2.8a), in the Born approximation the incident field freely propagates in the unperturbed structure. Supposing that the anisotropic scattering slab is homogeneous, the solution of Eq. (2.8a) for $0 \leq z \leq d$ reads

$$\psi_i(z) = \exp(ik_0 D_i z)\,\psi_i(0)\,, \tag{4.1}$$

where D_i is the propagation matrix (2.10) of the unperturbed anisotropic slab.

In the external regions $z < 0$ and $z > d$ the incident field, as we already discussed for the scattered Fourier components, is represented by a linear superposition of the four plane waves that are eigenvectors of the external D_i matrix, obtained by setting $\eta_{\alpha\beta} = n_1^2 \delta_{\alpha\beta}$ in (2.10). Using the same conventions as for the scattered fields, these eigenvectors $\psi_i^{(n)}$ $(n = 1, \ldots, 4)$ are given by Eqs. (3.2) and (3.3) with p_i and q_i replacing p and q, respectively. Therefore we make the decomposition, corresponding to the ϕ representation defined in [15],

$$\psi_i(z) = \sum_{n=1}^{4} \psi_i^{(n)} f_n(z) = T_i \phi_i(z)\,, \tag{4.2}$$

where T_i is the 4×4 matrix whose columns are given by the eigenvectors $\psi_i^{(n)}$, and $\phi_i(z)$ is the column vector containing the amplitudes $f_n(z)$. Now in the upper isotropic region the transmitted field must be represented by only forward solutions; therefore, with our conventions $\phi_i(d^+) = \phi_i^{(t)}$ has the third and fourth elements equal to zero. In the lower isotropic region, instead, the total field is represented by the sum of an incident and a reflected part, so that we can put $\phi_i(0^-) = \phi_i^{(i)} + \phi_i^{(r)}$, where $\phi_i^{(i)}$ is a given incident vector having the third and fourth elements equal to zero and $\phi_i^{(r)}$ is the reflected unknown field, whose first and second rows are equal to zero. We can now group the reflected and transmitted fields in a single vector by putting

$$\phi_i^{(r)} = P_r\, \phi_i^{(rt)}\,, \tag{4.3a}$$

$$\phi_i^{(t)} = P_t\, \phi_i^{(rt)}\,, \tag{4.3b}$$

where P_r (P_t) is the 4×4 diagonal projector matrix whose first two diagonal elements are equal to zero (one) and last two diagonal elements are equal to one (zero). From the continuity of the transverse components of the electromagnetic field across the dielectric boundaries and according to Eqs. (4.1)–(4.3) we then obtain

$$\phi_i^{(rt)} = W_i\, \phi_i^{(i)}\,, \tag{4.4}$$

where

$$W_i = [P_t - S_i P_r]^{-1} S_i\,, \tag{4.5a}$$

$$S_i = T_i^{-1} \exp(ik_0 D_i d)\, T_i\,. \tag{4.5b}$$

In this way the total incident field inside the scattering slab is given by

$$\psi_i(z) = \exp(ik_0 D_i z)\, T_i\, [P_r\, W_i + I]\, \phi_i^{(i)}\,, \tag{4.6}$$

with I being the 4×4 identity matrix.

Let us now go on to the scattered field. The general solution of Eq. (2.8b) inside the scattering medium reads

$$\psi_f(z) = \exp(ik_0 D_f z)\left[\psi_f(0) + ik_0 \int_0^z \exp(-ik_0 D_f z') \times G(z')\psi_i(z')\, dz'\right]\,, \tag{4.7}$$

where again we have considered the case of a homogeneous anisotropic slab. Similarly to Eqs. (4.2) and (3.1), we now decompose the scattered field in terms of the eigenvectors of the external homogeneous media

$$\psi_f(z) = \sum_{n=1}^{4} \psi_f^{(n)} c_n(z) = T_f \phi_f(z)\,, \tag{4.8}$$

where T_f is the 4×4 matrix whose columns are given by the eigenvectors $\psi_f^{(n)}$, and $\phi_f(z)$ is the column vector containing the amplitudes $c_n(z)$. The boundary conditions on the scattered field are that for $z = 0^-$ ($z = d^+$) the solution consists only of backwards (forwards) waves, so that we can introduce a column vector $\phi_f^{(rt)}$ such that

$$\phi_f(0^-) = P_r\, \phi_f^{(rt)}\,, \tag{4.9a}$$

$$\phi_f(d^+) = P_t\, \phi_f^{(rt)}\,. \tag{4.9b}$$

Finally, by putting together Eqs. (4.6)–(4.9) and using the continuity of the transverse components of the scattered field across the dielectric boundaries, we obtain the amplitudes of the scattered waves in the form

$$\phi_f^{(rt)} = \Gamma\, \phi_i^{(i)}\,, \tag{4.10}$$

where Γ is the scattering matrix

$$\Gamma = ik_0\, [T_f\, P_t - \exp(ik_0 D_f d)\, T_f\, P_r]^{-1} \exp(ik_0 D_f d) \times \left[\int_0^d \exp(-ik_0 D_f z)\, G(z)\, \exp(ik_0 D_i z)\, dz\right] \times T_i\, [P_r W_i + I]\,. \tag{4.11}$$

The scattering cross sections, i.e., the ratios between the scattered power per unit solid angle and the incident power per unit surface orthogonal to the propagation direction for given polarizations of the incident and scattered fields, are now readily evaluated on the basis of Eq. (3.10) and taking into account the fact that the incident proper waves have been normalized to unit power flux along the z direction and their Poynting vector is in the direction of propagation,

$$\frac{d\sigma}{d\Omega} = \lambda^2 n_1^2 \cos\vartheta_i \cos\vartheta_f |\Gamma_{mn}|^2\,, \tag{4.12}$$

where Γ_{mn} are the elements of the scattering matrix (4.11) computed for the transverse scattering wave vector (3.7) and $m = 1$ ($m = 2$) for TE (TM) forward scattering, $m = 3$ ($m = 4$) for TE (TM) backward scattering, while $n = 1$ ($n = 2$) for TE (TM) incident polarization. We recall that, with our conventions, the backward scattering cross sections obtained from Eq. (4.12) for $m = 3,4$ for given polar angles (ϑ_f, φ_f) correspond to the scattering direction $(\pi - \vartheta_f, \varphi_f)$.

V. COMPARISON WITH THE CLASSICAL FORMULAS

In order to compare our approach with the classical formulas for light scattering, obtained by making use of Green's propagator methods [5,6,10,11,18], we work out a few simple examples.

Let us begin with the simplest situation, namely the case in which the unperturbed scattering medium is isotropic, with the same index of refraction n_1 as the external homogeneous media, in such a way that no reflections occur at the boundary between the scattering and the external media. The exponential matrices describing the free propagation of the incident and scattered waves in the scattering region can then be expressed in terms of the matrices T_i and T_f, respectively, whose columns, as we already discussed, contain the proper waves (3.2) of the isotropic medium

$$\exp(ik_0 D_\alpha z) = T_\alpha \exp(ik_0 \Lambda_\alpha z)\, T_\alpha^{-1} \qquad (\alpha = i, f) , \tag{5.1}$$

where Λ_i (Λ_f) is the diagonal matrix whose diagonal elements are the eigenvalues γ_n ($n = 1, \ldots, 4$) of the matrix D_i (D_f), given by Eqs. (3.4) with p_i and q_i (p_f and q_f) replacing p and q, respectively. The matrices T_i^{-1} and T_f^{-1} are readily evaluated with the help of the orthonormality of the proper waves [15], and, with the definitions (3.3), read

$$T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} su & s/u & -cu & -c/u \\ cn_1/u & cu/n_1 & sn_1/u & su/n_1 \\ su & -s/u & -cu & c/u \\ -cn_1/u & cu/n_1 & -sn_1/u & su/n_1 \end{pmatrix} . \tag{5.2}$$

Using these expressions, according to Eqs. (4.12) and (4.11), the scattering cross sections can be cast in the form

$$\frac{d\sigma}{d\Omega} = \pi^2 \lambda^{-4} |\hat{\mathbf{f}} \cdot \delta\overleftrightarrow{\epsilon}(\vec{q}, t) \cdot \hat{\imath}|^2 , \tag{5.3}$$

where $\hat{\imath}$ ($\hat{\mathbf{f}}$) is the unit vector parallel to the incident (scattered) electric field and $\delta\overleftrightarrow{\epsilon}(\vec{q}, t)$ is the Fourier transform, computed for the scattering wave vector $\vec{q}$, of the fluctuations of the dielectric tensor, whose components are given by

$$\delta\epsilon_{\alpha\beta}(\vec{q}, t) = \int \delta\epsilon_{\alpha\beta}(\vec{r}, t) \exp(-i\vec{q} \cdot \vec{r})\, d\vec{r} \qquad (\alpha, \beta = x, y, z) . \tag{5.4}$$

Equation (5.3) coincides, as it should be, with the formula obtained by Green's propagator methods [5].

As a next richer example, we consider the case in which the scattering medium is an uniaxial anisotropic crystal, in order to test the corrections introduced by the misalignment between the Poynting vector and the wave vector directions inside an anisotropic crystal, and the transformations of solid angles between the internal and external media that are used to convert the internal scattering cross sections, computed with Green's propagator techniques, to the externally measured quantities.

Precisely, for the sake of definiteness, we suppose that the scattering medium is a homogeneously planarly oriented nematic liquid crystal undergoing small thermal fluctuations [19]. The geometry that we consider is shown in Fig. 1. The components of the relative dielectric tensor (2.2) inside the liquid crystal sample are given by

$$\epsilon_{\alpha\beta} = (n_e^2 - n_o^2)\, n_\alpha n_\beta + n_o^2 \delta_{\alpha\beta} , \tag{5.5}$$

n_o (n_e) being the ordinary (extraordinary) index of refraction, and n_α the Cartesian components of the nematic director $\hat{\mathbf{n}}$, a unit vector that specifies the local optical axis direction. For small thermal fluctuations around the homogeneous planar orientation $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, we can put [19]

$$\hat{\mathbf{n}} = \hat{\mathbf{x}} + n_y(\vec{r}, t)\, \hat{\mathbf{y}} + n_z(\vec{r}, t)\, \hat{\mathbf{z}} , \tag{5.6}$$

where, in the (x, z) plane, $n_y(\vec{r}, t)$ [$n_z(\vec{r}, t)$] is the small amplitude twist-bend (splay-bend) fluctuation component. Therefore, the nonzero components of the unperturbed relative dielectric tensor inside the sample are given by

$$\eta_{xx} = n_e^2 , \tag{5.7a}$$

$$\eta_{yy} = \eta_{zz} = n_o^2 , \tag{5.7b}$$

while, neglecting the products of the fluctuations, the nonzero components of the dielectric tensor fluctuations are

$$\delta\epsilon_{xy}(\vec{r}, t) = \delta\epsilon_{yx}(\vec{r}, t) = (n_e^2 - n_o^2)\, n_y(\vec{r}, t) , \tag{5.8a}$$

$$\delta\epsilon_{xz}(\vec{r}, t) = \delta\epsilon_{zx}(\vec{r}, t) = (n_e^2 - n_o^2)\, n_z(\vec{r}, t) . \tag{5.8b}$$

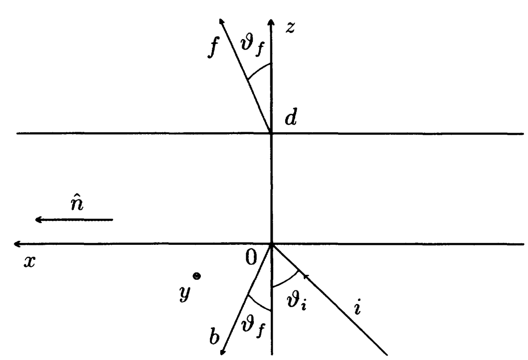

FIG. 1. Scattering geometry for the planarly oriented nematic liquid crystal cell. The sample is confined between the planes $z = 0$ and $z = d$ with an unperturbed director $\hat{\mathbf{n}}$ parallel to the x axis. The scattering occurs in the (x, z) plane, ϑ_i (ϑ_f) being the incidence (scattering) angle. i is the incident plane wave direction, f (b) the forwards (backwards) scattering direction.

The analogous of Eq. (5.1) now reads

$$\exp(ik_0 D_\alpha z) = T'_\alpha \exp(ik_0 \Lambda'_\alpha z)\, T'^{-1}_\alpha \qquad (\alpha = i, f) , \tag{5.9}$$

where the matrices T' contain the normalized proper modes of the unperturbed uniaxial crystal, that in our geometry are again given by TE and TM modes. With the same ordering conventions used for the external media, we have, for scattering in the (x, z) plane,

$$T' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_m & 0 & -u_m \\ 0 & u_m^{-1} & 0 & u_m^{-1} \\ -u_e^{-1} & 0 & -u_e^{-1} & 0 \\ -u_e & 0 & u_e & 0 \end{pmatrix} , \tag{5.10a}$$

$$T'^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -u_e & -u_e^{-1} \\ u_m^{-1} & u_m & 0 & 0 \\ 0 & 0 & -u_e & u_e^{-1} \\ -u_m^{-1} & u_m & 0 & 0 \end{pmatrix} , \tag{5.10b}$$

with

$$u_e = \sqrt[4]{n_o^2 - n_1^2 \sin^2 \vartheta} , \tag{5.11a}$$

$$u_m = \frac{\sqrt[4]{n_o^2 - n_1^2 \sin^2 \vartheta}}{\sqrt{n_o n_e}} , \tag{5.11b}$$

and $\vartheta = \vartheta_i$ ($\vartheta = \vartheta_f$) for the incidence (scattering) matrices T'_i and T'^{-1}_i (T'_f and T'^{-1}_f). As it is obvious, the T matrices of the external isotropic media in this scattering geometry are given by the same expressions (5.10) with $n_o = n_e = n_1$ in Eqs. (5.11). The diagonal elements of the diagonal matrices Λ' in (5.9) are the eigenvalues of the Berreman's matrices D in our uniaxial medium

$$\Lambda'_{11} \equiv \gamma_{\rm TE}(\vartheta) = \sqrt{n_o^2 - n_1^2 \sin^2 \vartheta} , \tag{5.12a}$$

$$\Lambda'_{22} \equiv \gamma_{\rm TM}(\vartheta) = \frac{n_e}{n_o} \sqrt{n_o^2 - n_1^2 \sin^2 \vartheta} , \tag{5.12b}$$

$$\Lambda'_{33} = -\gamma_{\rm TE}(\vartheta) , \tag{5.12c}$$

$$\Lambda'_{44} = -\gamma_{\rm TM}(\vartheta) , \tag{5.12d}$$

again with $\vartheta = \vartheta_i$ ($\vartheta = \vartheta_f$) for the incidence (scattering) matrix Λ'_i (Λ'_f). Substituting Eqs. (5.7)–(5.12) in Eqs. (4.11) and (2.12) we then obtain the scattering matrix

$$\Gamma = ik_0 \left[T_f\, P_t - T'_f \exp(ik_0 \Lambda'_f d)\, T'^{-1}_f\, T_f\, P_r \right]^{-1} T'_f \times \exp(ik_0 \Lambda'_f d)\, R\, T'^{-1}_i T_i \left[P_r W_i + I \right] , \tag{5.13}$$

where, of course, W_i is given by (4.5a) but with

$$S_i = T_i^{-1}\, T'_i \exp(ik_0 \Lambda'_i d)\, T'^{-1}_i\, T_i , \tag{5.14}$$

and R is the matrix

$$R = \frac{1}{2} \begin{pmatrix} 0 & -C n_y^{(\rm TM^+, TE^+)} & 0 & C n_y^{(\rm TM^-, TE^+)} \\ -D n_y^{(\rm TE^+, TM^+)} & -(A+B) n_z^{(\rm TM^+, TM^+)} & -D n_y^{(\rm TE^-, TM^+)} & (A-B) n_z^{(\rm TM^-, TM^+)} \\ 0 & C n_y^{(\rm TM^+, TE^-)} & 0 & -C n_y^{(\rm TM^-, TE^-)} \\ -D n_y^{(\rm TE^+, TM^-)} & (A-B) n_z^{(\rm TM^+, TM^-)} & -D n_y^{(\rm TE^-, TM^-)} & -(A+B) n_z^{(\rm TM^-, TM^-)} \end{pmatrix} , \tag{5.15}$$

with

$$A = \left(\frac{n_e^2}{n_o^2} - 1 \right) \sqrt[4]{\frac{n_o^2 - n_1^2 \sin^2 \vartheta_i}{n_o^2 - n_1^2 \sin^2 \vartheta_f}}\; n_1 \sin \vartheta_f , \tag{5.16a}$$

$$B = \left(\frac{n_e^2}{n_o^2} - 1 \right) \sqrt[4]{\frac{n_o^2 - n_1^2 \sin^2 \vartheta_f}{n_o^2 - n_1^2 \sin^2 \vartheta_i}}\; n_1 \sin \vartheta_i , \tag{5.16b}$$

$$C = \frac{(n_e^2 - n_o^2)}{\sqrt{n_o n_e}} \sqrt[4]{\frac{n_o^2 - n_1^2 \sin^2 \vartheta_i}{n_o^2 - n_1^2 \sin^2 \vartheta_f}} , \tag{5.16c}$$

$$D = \frac{(n_e^2 - n_o^2)}{\sqrt{n_o n_e}} \sqrt[4]{\frac{n_o^2 - n_1^2 \sin^2 \vartheta_f}{n_o^2 - n_1^2 \sin^2 \vartheta_i}} ; \tag{5.16d}$$

the symbol $n_y^{(\rm TM^+, TE^+)}$ represents the amplitude of the Fourier component of the twist-bend fluctuation mode corresponding to a forward incident TM wave and a forward scattered TE wave, and similarly for the other coefficients. According to (5.12), for $\alpha = x, y$ and $K, L =$ TE,TM these terms are explicitly given by

$$n_\alpha^{(K^\pm, L^\pm)} = \int_0^d n_\alpha(p_f - p_i, t) \times \exp\{ ik_0 [\pm \gamma_K(\vartheta_i) \mp \gamma_L(\vartheta_f)] z \}\, dz , \tag{5.17}$$

where the first (last) pair of upper and lower signs on the left hand side must be concurrently selected with the first (last) pair of upper and lower signs on the right hand side, and

$$n_\alpha(p, t) = \lambda^{-2} \int n_\alpha(\vec{r}, t) \exp(-ik_0 p x)\, dx\, dy . \tag{5.18}$$

Equation (5.15) shows that the splay-bend fluctuation modes give rise to scattering only for TM incident and scattered waves, while the twist-bend modes only for TM (TE) incident and TE (TM) scattered fields.

The matrices that multiply R on the left and right in the scattering matrix (5.13) represent the effect of the reflections of the scattered and incident fields, respectively, at the boundaries. In principle they could be determined analytically, but in practice the resulting expressions are far too cumbersome to be useful. Therefore we neglect the reflections by assuming, for $\alpha = i, f$, $T'_\alpha = T_\alpha$ in Eqs. (5.13) and (5.14). With this approximation, the nonzero scattering cross sections result

$$\frac{d\sigma^{(\mathrm{TE},\mathrm{TM}^+)}}{d\Omega} = \pi^2 n_1^2 |n_y^{(\mathrm{TE}^+,\mathrm{TM}^+)}|^2 \cos\vartheta_i \cos\vartheta_f \frac{(n_e^2-n_o^2)^2}{n_o n_e} \sqrt{\frac{n_o^2 - n_1^2\sin^2\vartheta_f}{n_o^2 - n_1^2\sin^2\vartheta_i}} , \tag{5.19a}$$

$$\frac{d\sigma^{(\mathrm{TE},\mathrm{TM}^-)}}{d\Omega} = \pi^2 n_1^2 |n_y^{(\mathrm{TE}^+,\mathrm{TM}^-)}|^2 \cos\vartheta_i \cos\vartheta_f \frac{(n_e^2-n_o^2)^2}{n_o n_e} \sqrt{\frac{n_o^2 - n_1^2\sin^2\vartheta_f}{n_o^2 - n_1^2\sin^2\vartheta_i}} , \tag{5.19b}$$

$$\frac{d\sigma^{(\mathrm{TM},\mathrm{TE}^+)}}{d\Omega} = \pi^2 n_1^2 |n_y^{(\mathrm{TM}^+,\mathrm{TE}^+)}|^2 \cos\vartheta_i \cos\vartheta_f \frac{(n_e^2-n_o^2)^2}{n_o n_e} \sqrt{\frac{n_o^2 - n_1^2\sin^2\vartheta_i}{n_o^2 - n_1^2\sin^2\vartheta_f}} , \tag{5.19c}$$

$$\frac{d\sigma^{(\mathrm{TM},\mathrm{TE}^-)}}{d\Omega} = \pi^2 n_1^2 |n_y^{(\mathrm{TM}^+,\mathrm{TE}^-)}|^2 \cos\vartheta_i \cos\vartheta_f \frac{(n_e^2-n_o^2)^2}{n_o n_e} \sqrt{\frac{n_o^2 - n_1^2\sin^2\vartheta_i}{n_o^2 - n_1^2\sin^2\vartheta_f}} , \tag{5.19d}$$

$$\frac{d\sigma^{(\mathrm{TM},\mathrm{TM}^+)}}{d\Omega} = \pi^2 n_1^4 |n_z^{(\mathrm{TM}^+,\mathrm{TM}^+)}|^2 \cos\vartheta_i \cos\vartheta_f \left(\frac{n_e^2}{n_o^2}-1\right)^2 \frac{\left[\sin\vartheta_f\sqrt{n_o^2-n_1^2\sin^2\vartheta_i}+\sin\vartheta_i\sqrt{n_o^2-n_1^2\sin^2\vartheta_f}\right]^2}{\sqrt{[n_o^2-n_1^2\sin^2\vartheta_f]\,[n_o^2-n_1^2\sin^2\vartheta_i]}} , \tag{5.19e}$$

$$\frac{d\sigma^{(\mathrm{TM},\mathrm{TM}^-)}}{d\Omega} = \pi^2 n_1^4 |n_z^{(\mathrm{TM}^+,\mathrm{TM}^-)}|^2 \cos\vartheta_i \cos\vartheta_f \left(\frac{n_e^2}{n_o^2}-1\right)^2 \frac{\left[\sin\vartheta_f\sqrt{n_o^2-n_1^2\sin^2\vartheta_i}-\sin\vartheta_i\sqrt{n_o^2-n_1^2\sin^2\vartheta_f}\right]^2}{\sqrt{[n_o^2-n_1^2\sin^2\vartheta_f]\,[n_o^2-n_1^2\sin^2\vartheta_i]}} , \tag{5.19f}$$

where the symbol $d\sigma^{(\mathrm{TE},\mathrm{TM}^+)}/d\Omega$ stands for the differential scattering cross section for an incident TE wave and a forwards scattered TM wave, and similarly for the other cases.

Let us now consider the differential scattering cross sections for a homogeneous anisotropic medium computed using the Green's propagator [6,18,20]

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{in}} = \frac{\pi^2\lambda^{-4} n_f}{n_i\cos\delta_i\,\cos^2\delta_f} |\hat{\mathrm{f}}\cdot\delta\overleftrightarrow{\epsilon}(\vec{q},t)\cdot\hat{\imath}|^2 , \tag{5.20}$$

where n_i (n_f) is the index of refraction for the incident (scattered) field, δ_i (δ_f) is the angle formed between the electric field $\vec{E}$ and the electric displacement $\vec{D}$ of the incident (scattered) wave, that coincides with the angle formed between the wave vector and the Poynting vector directions, and the other symbols have the same meaning as in (5.3). In Eq. (5.20) the solid angles are referred to the scattered wave vectors directions.

In order to compare (5.20), that refers to the inside of the scattering medium, with (5.19), that instead refer to the outside, we must introduce two corrections for the incident and scattered fields. First of all we must transform, according to Snell's law, the internal angles about the wave vector direction into the external ones [6,18]

$$\frac{d\Omega_{\mathrm{in}}}{d\Omega_{\mathrm{out}}} = \frac{n_1^2\cos\vartheta_f\cos\delta_f}{n_f^2\cos\psi_f} , \tag{5.21}$$

where ψ_f is the angle formed between the Poynting vector of the scattered wave and the z axis. Lastly, we must take into account that (5.20) refers to an incident power flux P_i^{in} computed inside the anisotropic medium, while (5.19) to the incident power flux P_i^{out} outside the anisotropic medium. Now it is easy to understand that the way in which we have neglected the reflections of the incident field is equivalent to supposing that the z components of the Poynting vectors inside and outside the anisotropic medium are equal; in this approximation, since the external Poynting vector is directed along the wave vector, we have

$$\frac{P_i^{\mathrm{in}}}{P_i^{\mathrm{out}}} = \frac{\cos\vartheta_i}{\cos\psi_i} , \tag{5.22}$$

being ψ_i the angle formed between the Poynting vector of the incident wave and the z axis. Then, from (5.20)–(5.22), the external scattering cross sections read

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{out}} = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{in}} \frac{d\Omega_{\mathrm{in}}}{d\Omega_{\mathrm{out}}}\frac{P_i^{\mathrm{in}}}{P_i^{\mathrm{out}}} = \pi^2\lambda^{-4}\frac{n_1^2\cos\vartheta_i\cos\vartheta_f}{n_i\,\cos\delta_i\,\cos\psi_i\,n_f\cos\delta_f\,\cos\psi_f}|\hat{\mathrm{f}}\cdot\delta\overleftrightarrow{\epsilon}(\vec{q},t)\cdot\hat{\imath}|^2 , \tag{5.23}$$

and a straightforward calculation shows that they coincide with (5.19).

This example demonstrates how Eq. (4.12) automatically takes into account all the solid angle transformations between the internal and external media, and the complications arising from the misalignment between the wave vector and the Poynting vector directions inside an anisotropic medium [6–9]. Moreover, it also fully incorporates all the boundary reflections and transmissions for both the incident and the scattered fields. To see how these affect the results, we show, in Figs. 2 and 3, the exact and the approximate normalized scattering cross sections due to the splay-bend fluctuation modes. For $\alpha, \beta, \gamma = +, -$ these are defined as

$$\rho_{\alpha\beta}^{(\gamma)} = \frac{1}{\left|n_z^{(\mathrm{TM}^\alpha,\mathrm{TM}^\beta)}\right|^2} \frac{d\sigma^{(\mathrm{TM},\mathrm{TM}^\gamma)}}{d\Omega} , \tag{5.24}$$

where the scattering cross section is computed by putting to zero all the Fourier components of the fluctuations different from $n_z^{(\mathrm{TM}^\alpha,\mathrm{TM}^\beta)}$.

The direct contributions to the forward and backward normalized scattering cross sections, due to the Fourier components of the fluctuations directly coupling the forward incident beam inside the anisotropic medium with the forward and backward, respectively, internal scattered fields, are shown in Figs. 2(a) and 2(b), respectively. The solid lines are the exact curves, numerically computed on the basis of Eq. (5.13), while the dashed ones are the approximate expressions given by Eqs. (5.19e) and (5.19f). It is apparent that even for a relatively high mismatch between the internal and the external indices of refraction, the approximate analytical expressions give a fairly good answer; the biggest differences with the exact curves are mainly due to the interference fringes created by the multiple reflections at the boundaries. These figures also show that, even for the exact curves, $\rho_{++}^{(+)}(\vartheta_i, \vartheta_f) = \rho_{+-}^{(-)}(\vartheta_i, -\vartheta_f)$, a reciprocity relation that is linked to the symmetries of the Berreman's propagation matrices in our lossless unperturbed structure.

The other effect induced by the boundary reflections is the mixing of the various longitudinal Fourier components. In fact, as shown in Figs. 2(c) and 2(d), the Fourier component of the fluctuations that couples the forward incident beam with the forward (respectively backward) scattered wave, also gives rise to a contribution to the backward (respectively forward) scattering, due to the reflections of the scattered fields. As one sees from the figures, for high scattering angles these contributions can be comparable or even dominant with respect to the direct terms. The reciprocity relation noted in the previous two cases holds true also in this case, as $\rho_{++}^{(-)}(\vartheta_i, \vartheta_f) = \rho_{+-}^{(+)}(\vartheta_i, -\vartheta_f)$. Figures 3(a) and 3(b) show, instead, smaller contribution to the forward (respectively backward) scattering due to reflections of the incident field; here too we have the reciprocity condi-

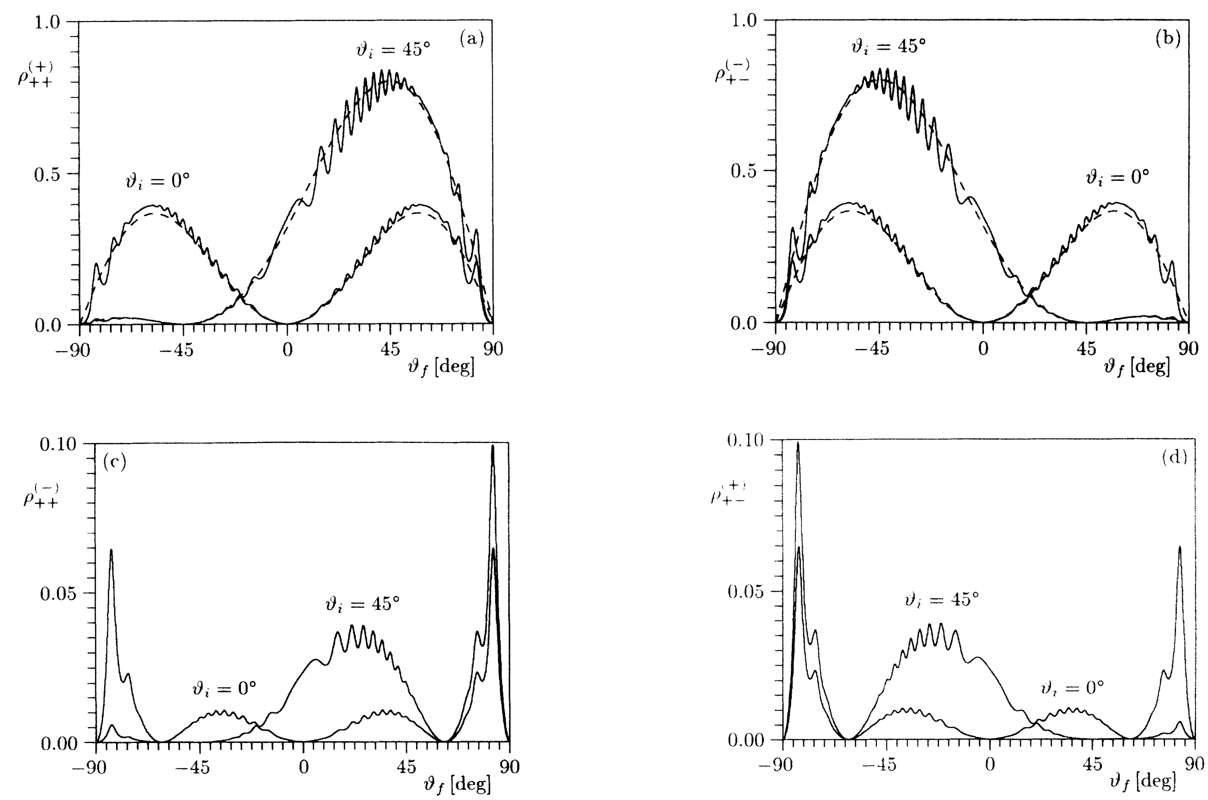

FIG. 2. Normalized scattering cross sections for TM incident and scattered polarizations for the geometry shown in Fig. 1 as a function of the scattering angle ϑ_f with incidence angles equal to $\vartheta_i = 0°$ and $\vartheta_i = 45°$. The ordinary (extraordinary) index of refraction is $n_o = 1.5$ ($n_e = 1.7$); the index of refraction of the external media is $n_1 = 1$; the thickness of the sample is equal to $d = 20\lambda$. (a) Direct contribution to the forward scattering. The solid lines are the exact curves, the dashed ones the approximate analytical expressions obtained by neglecting the reflections and transmissions at the boundaries. (b) Same as (a) but for the backward scattering. (c) Indirect contribution to the backward scattering due to the reflections of the scattered waves. (d) Same as (c) but for the forward scattering.

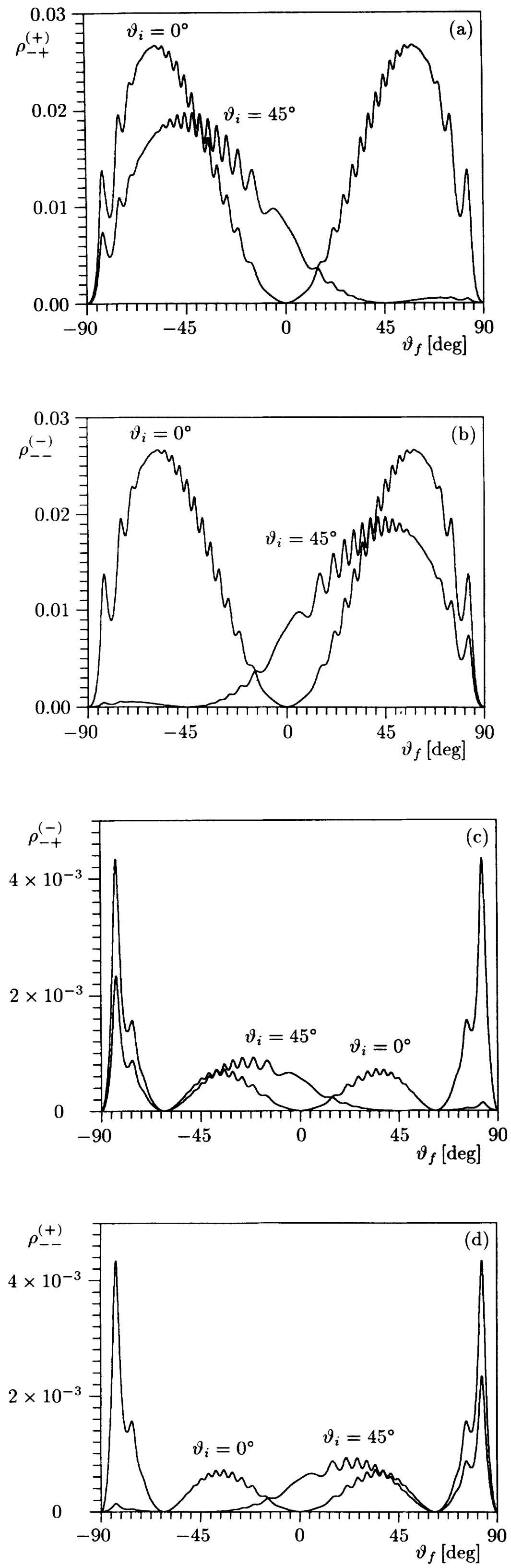

FIG. 3. Same as Fig. 2 but for the longitudinal Fourier components giving rise to indirect contributions to the forward [(a) and (d)] and backward [(b) and (c)] scattering due to the reflections of the incident field [(a) and (b)] and of both the incident and the scattered fields [(c) and (d)].

tion $\rho_{-+}^{(+)}(\vartheta_i, \vartheta_f) = \rho_{--}^{(-)}(\vartheta_i, -\vartheta_f)$. Finally the smallest contributions, due to the reflections of both the incident and the scattered fields, are shown in Figs. 3(c) and 3(d); even in this case we have $\rho_{-+}^{(-)}(\vartheta_i, \vartheta_f) = \rho_{--}^{(+)}(\vartheta_i, -\vartheta_f)$. Similar results are obtained for the depolarized scatterings induced by the twist-bend fluctuation modes.

VI. ARBITRARILY STRATIFIED MEDIA

A further advantage of our approach, with respect to the classical Green's propagator calculations, is that it can be immediately generalized to the most arbitrary variation along the longitudinal direction, stepwise or continuous, of the dielectric tensor of the unperturbed structure, both in the scattering and in the external regions. This is useful for the correct analysis of the actual situations, in which the external media are usually stratified, and for the study of distorted scattering media [13].

It is easily recognized that, when the outermost media are isotropic, the most general case can be always represented by a medium with a continuous variation of the unperturbed dielectric tensor in the region $0 \leq z \leq d$, bounded by two homogeneous isotropic media for $z < 0$ and $z > d$ having indices of refraction n_i and n_f, respectively: in fact, when the transparent external media are stratified, we can consider the intermediate transparent slabs as a part of the varying scattering medium having zero amplitudes for the fluctuations of the dielectric tensor.

Thus, by introducing the evolution matrices $U_i(z)$ and $U_f(z)$ that are solutions of the unperturbed propagation equations for $0 \leq z \leq d$,

$$\frac{dU_\alpha(z)}{dz} = ik_0\, D_\alpha(z)\, U_\alpha(z)\ , \quad U_\alpha(0) = I \quad (\alpha = i, f)\ , \tag{6.1}$$

Eqs. (4.11) and (4.12) respectively become

$$\Gamma = ik_0\, [T_f(d)\, P_t - U_f(d)\, T_f(0)\, P_r]^{-1}\, U_f(d) \times \left[\int_0^d U_f^{-1}(z)\, G(z)\, U_i(z)\, dz\right] T_i(0)\, [P_r W_i + I]\ , \tag{6.2}$$

and

$$\frac{d\sigma}{d\Omega} = \lambda^2 n^2 \cos\vartheta_i \cos\vartheta_f |\Gamma_{mn}|^2\ , \tag{6.3}$$

where $n = n_f$ ($n = n_i$) for forward (backward) scattering; the matrix W_i is given by (4.5a) but with

$$S_i = T_i^{-1}(d)\, U_i(d)\, T_i(0)\ ; \tag{6.4}$$

the matrices $T_i(0)$ and $T_f(0)$ contain the proper waves for the incident and scattered fields in the incidence isotropic medium having index of refraction n_i, and, similarly, $T_i(d)$ and $T_f(d)$ in the transmitted medium having index of refraction n_f. Finally, of course, in Eqs. (2.6) and (3.7) we must substitute n_1 with n_i and n_f, respectively.

VII. CONCLUSIONS

We have developed an alternative approach, based on a suitable generalization of the propagation equations for transparent stratified anisotropic media, to exactly calculate, in the Born approximation, the scattering cross sections in a generic anisotropic stratified medium.

By working out a few significant examples we have also shown that, with appropriate approximations that amount to neglecting the reflections and transmissions of the incident and scattered waves at the dielectric boundaries, from our formulas we recover the already known results obtained with a much more delicate analysis based on the study of the asymptotic behavior of the Green's propagators for anisotropic media and a separate careful exam of the effects introduced by the boundaries [6]. With our formalism, instead, all the boundary effects and the corrections introduced by the birefringence of the scattering medium are automatically taken into account. Moreover, the resulting formulas can be straightforwardly computed numerically, since they involve very simple algebraic matrix manipulations: in this respect a comparison between the exact and the approximate expressions has been presented, showing how the boundary conditions alter the scattering cross sections and mix the various longitudinal Fourier components of the fluctuations.

Finally we have shown that our results allow one to tackle the case of a continuously distorted scattering medium, a situation that cannot be dealt with using the Green's propagator approach, in a way just as simple as for a homogeneous medium.

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